

# Classical Enhancement of Quantum Error-Correcting Codes

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We present a general formalism for quantum error-correcting codes that encode both classical and quantum information (the EACQ formalism). This formalism unifies the entanglement-assisted formalism and classical error correction, and includes encoding, error correction, and decoding steps such that the encoded quantum and classical information can be correctly recovered by the receiver. We formally define this kind of quantum code using both stabilizer and symplectic language, and derive the appropriate error-correcting conditions. We give several examples to demonstrate the construction of such codes.

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## I. INTRODUCTION

Since Shor proposed the first quantum error correction code (QECC) [21], research in this field has progressed rapidly. A broad theory of quantum error-correcting codes was created with the stabilizer formalism and its symplectic formulation [8, 11], that allow the systematic description of a large class of quantum error correction codes and their error-correcting properties. In this formulation, a QECC is defined to be a subspace fixed by a *stabilizer group*. At the same time, a construction of QECCs from classical error correction codes was proposed separately by Calderbank, Shor and Steane [9, 22], the so-called CSS construction. Later this was generalized to give a stronger connection between quantum codes and classical symplectic codes; however, it seemed that this connection between quantum coding theory and classical coding theory was not universal, since only certain symplectic codes possessed quantum equivalents.

More recent developments in quantum coding theory have led to the development of the operator quantum error correction formalism (OQECC) [1, 2, 14, 15, 16, 19, 20] and the entanglement-assisted quantum error correction formalism (EAQECC) [5, 6, 7]; moreover, it is possible to produce a unified formalism (EAOQECC) [13] that combines both OQECCs and EAQECCs. This formalism demonstrates that a broader connection exists between classical and quantum error correction theory. Good QECCs can be obtained by a generalized CSS construction from good classical codes. This opens the door, for example, to the construction of high-quality quantum codes from modern classical codes, such as Turbo and LDPC codes [12].

In this paper, we generalize this construction in a different way, by proposing new quantum codes that can be

used to transmit both classical and quantum information simultaneously. We call this scheme the entanglement-assisted, classically enhanced quantum error correction formalism, but throughout the paper it will be referred to simply as the EACQ formalism. The EACQ formalism can be considered a generalization EAQECCs, or as a unification of quantum and classical linear error correction codes. This unification also makes contact with results in quantum information theory, where bounds exist on the asymptotic transmission of simultaneous classical and quantum information, including the use of entanglement assistance. It is believed that these bounds are better than simple time-sharing between codes for transmitting quantum and classical information separately through a quantum channel [10]. It is our hope that it may be possible to construct classes of codes which achieve these rates in the limit of large block size.

This paper is organized as follows. We give a brief introduction of the EAQECC formalism using both the stabilizer and the symplectic language in section II. In section III, we formally define a quantum code (EACQ) that can transmit both classical and quantum information at the same time. Several properties of this kind of quantum code are also discussed in this section. We provide several examples in section IV, to demonstrate the usefulness of this formalism. We conclude in section V by examining some special cases, and arguing that the EACQ formalism is indeed a generalization and unification of quantum and classical coding theory.

## II. EAQECC

In this section, we will review entanglement-assisted quantum error correction using both stabilizer and symplectic language.

Let  $\mathcal{G}_n$  be the  $n$ -fold Pauli Group [18]. Every operator in  $\mathcal{G}_n$  has either eigenvalues  $\pm 1$  or  $\pm i$ . An  $[[n, q, d; e]]$  EAQECC is a quantum code that encodes  $q$  logical quantum bits (qubits) into  $n$  phys-

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ical qubits with the help of  $e$  maximally entangled pairs (ebits) shared between sender and receiver, and can correct up to  $\lfloor d/2 \rfloor$  single-qubit errors. Such an EAQECC is defined by a non-commuting group  $\mathcal{S}_Q = \langle \bar{Z}_1, \dots, \bar{Z}_s, \bar{Z}_{s+1}, \bar{X}_{s+1}, \dots, \bar{Z}_{s+e}, \bar{X}_{s+e} \rangle \subset \mathcal{G}_n$  of size  $2^{s+2e}$ , where  $s + e + q = n$ , and the generators  $\bar{Z}_i$  and  $\bar{X}_i$  satisfy the following commutation relations:

$$\begin{aligned} [\bar{Z}_i, \bar{Z}_j] &= 0 \quad \forall i, j \\ [\bar{X}_i, \bar{X}_j] &= 0 \quad \forall i, j \\ [\bar{X}_i, \bar{Z}_j] &= 0 \quad \forall i \neq j \\ \{\bar{X}_i, \bar{Z}_i\} &= 0 \quad \forall i. \end{aligned} \quad (1)$$

We define the isotropic subgroup  $\mathcal{S}_{Q,I}$  of  $\mathcal{S}_Q$  to be the subgroup generated by  $\{\bar{Z}_1, \dots, \bar{Z}_s\}$ ; it is of size  $2^s$ . Similarly, the symplectic subgroup  $\mathcal{S}_{Q,S}$  of  $\mathcal{S}_Q$  is of size  $2^{2e}$  and is generated by  $\{\bar{Z}_{s+1}, \bar{X}_{s+1}, \dots, \bar{Z}_{s+e}, \bar{X}_{s+e}\}$ . The isotropic subgroup  $\mathcal{S}_{Q,I}$  is Abelian; however, the symplectic subgroup  $\mathcal{S}_{Q,S}$  is not. We can easily construct an Abelian extension of  $\mathcal{S}_{Q,S}$  that acts on  $n + e$  qubits, by specifying the following generators:

$$\begin{aligned} \bar{Z}_1 &\otimes I, \\ &\vdots \\ \bar{Z}_s &\otimes I, \\ \bar{Z}_{s+1} &\otimes Z_1, \\ \bar{X}_{s+1} &\otimes X_1, \\ &\vdots \\ \bar{Z}_{s+e} &\otimes Z_e, \\ \bar{X}_{s+e} &\otimes X_e, \end{aligned}$$

where the first  $n$  qubits are on the side of the sender (Alice) and the extra  $e$  qubits are taken to be on the side of the receiver (Bob). The operators  $Z_i$  or  $X_i$  to the right of the tensor product symbol above is the Pauli operator  $Z$  or  $X$  acting on Bob's  $i$ -th qubit. The picture is that Alice and Bob initially share  $e$  ebits; Alice encodes her  $q$  qubits together with her halves of the  $e$  entangled pairs and  $s$  ancilla qubits. Bob's qubits are his halves of the  $e$  entangled pairs. Because this code assumes pre-existing entanglement between Alice and Bob, it is an entanglement-assisted quantum error-correcting code (EAQECC). We denote such an Abelian extension of the group  $\mathcal{S}_{Q,S}$  by  $\tilde{\mathcal{S}}_{Q,S}$ . This EAQECC can correct an error set  $\mathbf{E} \subset \mathcal{G}_n$  if for all  $E_1, E_2 \in \mathbf{E}$ ,  $E_2^\dagger E_1 \in \mathcal{S}_{Q,I} \cup (\mathcal{G}_n - N(\mathcal{S}_Q))$ , where  $N(\mathcal{S})$  is the normalizer of group  $\mathcal{S}$ .

Before we describe EAQECCs using the symplectic language, we need to first discuss some of the basic properties of the symplectic form which are relevant to the discussion that follows. The symplectic form of vectors in  $(\mathbb{Z}_2)^{2n}$  is useful for specifying Pauli operators on  $n$  qubits when the global phase may be ignored. We write a vector  $\mathbf{u} \in (\mathbb{Z}_2)^{2n}$  in symplectic form by splitting it into two vectors  $\mathbf{x}, \mathbf{z} \in (\mathbb{Z}_2)^n$  and writing it as follows:

$\mathbf{u} = (\mathbf{z}|\mathbf{x})$ . We define

$$N_{(\mathbf{z}|\mathbf{x})} \equiv Z^{z_1} X^{x_1} \otimes Z^{z_2} X^{x_2} \otimes \dots \otimes Z^{z_n} X^{x_n},$$

where  $z_r$  ( $x_r$ ) is the  $r$ -th element of  $\mathbf{z}$  ( $\mathbf{x}$ ). Thus a set of  $m$  Pauli-operators acting on  $n$  qubits may be specified by a matrix with  $m$  rows  $\mathbf{u}_i \in (\mathbb{Z}_2)^{2n}$ ,  $i = 1, 2, \dots, m$ . The symplectic product between two vectors is defined as

$$(\mathbf{z}|\mathbf{x}) \odot (\mathbf{z}'|\mathbf{x}') = \mathbf{z} \cdot \mathbf{x}'^T - \mathbf{x} \cdot \mathbf{z}'^T.$$

(Note that in the binary case, as here, subtraction is the same as addition.) Two Pauli operators  $N_{(\mathbf{z}|\mathbf{x})}$  and  $N_{(\mathbf{z}'|\mathbf{x}')}$  commute if and only if  $(\mathbf{z}|\mathbf{x}) \odot (\mathbf{z}'|\mathbf{x}') = 0$ .

Recall that the stabilizer  $\mathcal{S}_Q$  of an  $[[n, q; e]]$  EAQECC is generated by  $s + 2e$  elements. Therefore, it can be specified by an  $(s + 2e) \times 2n$  symplectic matrix,  $\hat{F}$ , which we will refer to as the *quantum parity check matrix* in this paper. Thus,

$$\mathcal{S}_Q = \{N_{\mathbf{u}} | \mathbf{u} \in \text{Rowspace}(\hat{F})\}, \quad (2)$$

where

$$\hat{F} = \begin{pmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_{s+e} \\ \mathbf{v}_{s+1} \\ \vdots \\ \mathbf{v}_{s+e} \end{pmatrix}. \quad (3)$$

In this matrix, the rows  $\mathbf{u}_1 \dots \mathbf{u}_{s+e}$  represent the generators  $\bar{Z}_1 \dots \bar{Z}_{s+e}$ , and the rows  $\mathbf{v}_{s+1} \dots \mathbf{v}_{s+e}$  represent  $\bar{X}_{s+1} \dots \bar{X}_{s+e}$ . The commutation relations in (1) translate to the following:

$$\begin{aligned} \mathbf{u}_i \odot \mathbf{u}_j &= 0 \quad \forall i, j \\ \mathbf{v}_i \odot \mathbf{v}_j &= 0 \quad \forall i, j \\ \mathbf{u}_i \odot \mathbf{v}_j &= 0 \quad \forall i \neq j \\ \mathbf{u}_i \odot \mathbf{v}_i &= 1 \quad \forall i. \end{aligned} \quad (4)$$

The isotropic subgroup  $\mathcal{S}_{Q,I}$  and the symplectic subgroup  $\mathcal{S}_{Q,S}$  can be rewritten as:

$$\begin{aligned} \mathcal{S}_{Q,I} &= \{N_{\mathbf{u}} | \mathbf{u} \in \text{Rowspace}(\hat{F}_I)\}, \\ \mathcal{S}_{Q,S} &= \{N_{\mathbf{u}} | \mathbf{u} \in \text{Rowspace}(\hat{F}_S)\}, \end{aligned}$$

up to an overall phase, where

$$\hat{F}_I = \begin{pmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_s \end{pmatrix}, \quad \hat{F}_S = \begin{pmatrix} \mathbf{u}_{s+1} \\ \vdots \\ \mathbf{u}_{s+e} \\ \mathbf{v}_{s+1} \\ \vdots \\ \mathbf{v}_{s+e} \end{pmatrix}. \quad (5)$$

We can now specify the error correcting condition in the symplectic formulation. This EAQECC can correct an error set  $\mathbf{E} \subset (\mathbb{Z}_2)^{2n}$  if for all  $\mathbf{e}_1, \mathbf{e}_2 \in \mathbf{E}$ , either  $\hat{F} \odot (\mathbf{e}_2 - \mathbf{e}_1) \neq 0$  or  $(\mathbf{e}_2 - \mathbf{e}_1) \in \text{Rowspace}(\hat{F}_I)$ .

### III. CLASSICALLY ENHANCED QUANTUM ERROR CORRECTION

In this section, we will present a new quantum code that can transmit both classical and quantum information at the same time.

#### A. The Stabilizer Formalism

We define an  $[[n, q : c, d; e]]$  entanglement-assisted, classically enhanced quantum error correction code (EACQ) to be a quantum code which encodes  $q$  logical qubits and  $c$  classical bits into  $n$  physical qubits with the help of  $e$  ebits. Our quantum information is given by the  $q$ -dimensional state  $|\phi\rangle \in (\mathcal{H}_2)^{\otimes q}$ , and our classical information  $i \in \{1, 2, \dots, 2^c\}$  is represented by a vector  $\mathbf{x}_i \in (\mathbb{Z}_2)^c$ . Here, we keep the subscript  $i$  in  $\mathbf{x}_i$  to remind the reader that  $\mathbf{x}_i$  is the binary expression of  $i$ . Let us denote the  $2^q$ -dimensional Hilbert space of the original qubits by  $\mathcal{H} \equiv (\mathcal{H}_2)^{\otimes q}$ , and the subspaces of the  $n$ -dimensional encoded states by  $\mathcal{C}^i$ . Our encoding operations  $\tilde{U}_{enc}^i : \mathcal{H} \rightarrow \mathcal{C}^i$  consist of appending the ancilla states  $|0\rangle^{\otimes s}$  and maximally entangled states  $|\Phi_+\rangle^{\otimes e}$ , where  $s + e + q = n$  and  $|\Phi_+\rangle \equiv \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ , to  $|\phi\rangle$  followed by performing the unitary  $U_i$ . Thus, our encoded states, or "codewords", are defined as

$$|\Psi_i\rangle \equiv U_i(|0\rangle^{\otimes s} \otimes |\Phi_+\rangle^{\otimes e} \otimes |\phi\rangle). \quad (6)$$

We require that  $\langle \Psi_i | \Psi_j \rangle = \delta_{ij}$  so that the classical information is perfectly retrievable.

**Theorem 1** *We specify an  $[[n, q : c, d; e]]$  EACQ by the pair of groups  $(\mathcal{S}_Q, \mathcal{S}_C)$ . The quantum stabilizer  $\mathcal{S}_Q = \langle \mathcal{S}_{Q,I}, \mathcal{S}_{Q,S} \rangle$  of the code is generated by  $s+2e-c$  elements:*

$$\begin{aligned} \mathcal{S}_{Q,I} &= \langle \overline{Z}_{c_1+1}, \overline{Z}_{c_1+2}, \dots, \overline{Z}_s \rangle, \\ \mathcal{S}_{Q,S} &= \langle \overline{Z}_{s+c_2+1}, \overline{X}_{s+c_2+1}, \dots, \overline{Z}_{s+e}, \overline{X}_{s+e} \rangle. \end{aligned} \quad (7)$$

*The classical stabilizer  $\mathcal{S}_C = \langle \mathcal{S}_{C,I}, \mathcal{S}_{C,S} \rangle$  of the code is generated by  $c$  elements:*

$$\begin{aligned} \mathcal{S}_{C,I} &= \langle \overline{Z}_1, \overline{Z}_2, \dots, \overline{Z}_{c_1} \rangle, \\ \mathcal{S}_{C,S} &= \langle \overline{Z}_{s+1}, \dots, \overline{Z}_{s+c_2}, \overline{X}_{s+1}, \dots, \overline{X}_{s+c_2} \rangle, \end{aligned} \quad (8)$$

where  $q + s + e = n$  and  $c_1 + 2c_2 = c$ , such that,  $\forall g_j \in \mathcal{S}_Q$ ,

$$g_j |\Psi_i\rangle = |\Psi_i\rangle, \quad (9)$$

and

$$g'_j |\Psi_i\rangle = (-1)^{x_{ij}} |\Psi_i\rangle, \quad (10)$$

where  $g'_j$  is the  $j$ -th element of the generator set of  $\tilde{\mathcal{S}}_C$ , which is the Abelian extension of  $\mathcal{S}_C$ , and  $x_{ij}$  is the  $j$ -th element of  $\mathbf{x}_i \in (\mathbb{Z}_2)^c$ .

**Proof** We begin with a canonical code that encodes the quantum information  $|\phi\rangle \in (\mathcal{H}_2)^{\otimes q}$  together with classical information  $\mathbf{x}_i$  in the following trivial way:

$$\begin{aligned} |\phi\rangle \xrightarrow{\mathbf{x}_i} |\psi_i\rangle &= (N_{(\mathbf{0}|\mathbf{x}_a)} |0\rangle^{\otimes c_1}) |0\rangle^{\otimes (s-c_1)} \\ &\quad \left[ \left( N_{(\mathbf{x}_{b_2}|\mathbf{x}_{b_1})} \otimes I \right) |\Phi_+\rangle^{\otimes c_2} \right] |\Phi_+\rangle^{\otimes e-c_2} |\phi\rangle, \end{aligned} \quad (11)$$

where  $\mathbf{x}_a \in (\mathbb{Z}_2)^{c_1}$  and  $\mathbf{x}_{b_1}, \mathbf{x}_{b_2} \in (\mathbb{Z}_2)^{c_2}$ , and  $I$  is the  $c_2 \times c_2$  identity acting on Bob's qubits. Instead of encoding  $\mathbf{x}_i$  as a whole, we separate  $\mathbf{x}_i$  into  $\mathbf{x}_a = x_{i1} \dots x_{ic_1}$ ,  $\mathbf{x}_{b_1} = x_{i,(c_1+1)} \dots x_{i,(c_1+c_2)}$ , and  $\mathbf{x}_{b_2} = x_{i,(c_1+c_2+1)} \dots x_{ic}$  such that  $c_1 + 2c_2 = c$ , and encode  $\mathbf{x}_{b_1}$  and  $\mathbf{x}_{b_2}$  using  $c_2$  pairs of maximally entangled states.  $\mathbf{x}_a$  Clearly, the set  $\{|\psi_i\rangle\}$  is stabilized by  $\mathcal{S}'_Q = \langle \mathcal{S}'_{Q,I}, \mathcal{S}'_{Q,S} \rangle$ , where

$$\begin{aligned} \mathcal{S}'_{Q,I} &= \langle Z_{c_1+1}, Z_{c_1+2}, \dots, Z_s \rangle, \\ \mathcal{S}'_{Q,S} &= \langle Z_{s+c_2+1}, X_{s+c_2+1}, \dots, Z_{s+e}, X_{s+e} \rangle. \end{aligned} \quad (12)$$

Now let  $\mathcal{S}'_C = \langle \mathcal{S}'_{C,I}, \mathcal{S}'_{C,S} \rangle$ , where

$$\begin{aligned} \mathcal{S}'_{C,I} &= \langle Z_1, \dots, Z_{c_1} \rangle, \\ \mathcal{S}'_{C,S} &= \langle Z_{s+1}, \dots, Z_{s+c_2}, X_{s+1}, \dots, X_{s+c_2} \rangle, \end{aligned} \quad (13)$$

and let  $\tilde{\mathcal{S}}'_C$  be the Abelian extension of  $\mathcal{S}'_C$ . Then it is easy to verify that

$$g'_j |\psi_i\rangle = (-1)^{x_{ij}} |\psi_i\rangle, \quad (14)$$

where  $g'_j$  is the  $j$ -th generator of  $\tilde{\mathcal{S}}'_C$ .

Since  $(\mathcal{S}'_Q, \mathcal{S}'_C)$  is isomorphic to  $(\mathcal{S}_Q, \mathcal{S}_C)$ , there exists an unitary  $U$  such that  $\mathcal{S}_Q = U \mathcal{S}'_Q U^\dagger$  and  $\mathcal{S}_C = U \mathcal{S}'_C U^\dagger$ . The codewords  $\{|\Psi_i\rangle\}$  can also be obtained by

$$U |\psi_i\rangle = |\Psi_i\rangle. \quad (15)$$

It is then easy to verify that (9) and (10) hold.  $\square$

Notice that  $\langle \mathcal{S}_Q, \mathcal{S}_C \rangle$  is the stabilizer of an  $[[n, q; e]]$  EAQECC code, and thus it fully specifies one of the codewords from (6),  $|\Psi_0\rangle$ . For  $c > 0$ , the additional codewords are just unitary transformations of  $|\Psi_0\rangle$ . Theorem 1 confirms that  $\mathcal{S}_C$  and  $\mathcal{S}_Q$  together are sufficient to fully specify the codewords.

Now that we have uniquely defined our code, we will consider the conditions that make a set of errors correctable, as well as the decoding procedure for a given set of correctable errors. We will consider here only error sets which are subsets of  $\mathcal{G}^n$ , since it has been shown that the ability to correct such a discrete error set implies the ability to correct any linear combination of errors in that set.

**Theorem 2** *A set of errors  $\mathbf{E} \subset \mathcal{G}^n$  is correctable if for all  $E_m, E_p \in \mathbf{E}$ ,  $E_m^\dagger E_p \in \langle \mathcal{S}_{Q,I}, \mathcal{S}_{C,I} \rangle \cup (\mathcal{G}^n - N(\mathcal{S}_Q))$ , where  $N(\mathcal{S})$  is the normalizer of group  $\mathcal{S}$ .*

**Proof** We consider the following different cases.

1. If  $E_m^\dagger E_p \in \mathcal{G}^n - N(\mathcal{S}_Q)$ , then by definition there is at least one element  $g_j \in \mathcal{S}_Q$  such that

$$[E_m^\dagger E_p, g_j] \neq 0.$$

Then we are guaranteed that  $E_m$  and  $E_p$  have different error syndromes on the set of codewords  $\{|\Psi_i\rangle\}$ . We can then perform a recovery operation based on the error syndrome. If it is determined that the error  $E_m$  occurred, the original codeword may be recovered by simply performing the unitary  $E_m$  since  $E_m \in \mathcal{G}^n$ .

2. If  $E_m^\dagger E_p \in N(\mathcal{S}_Q)$ , there are three cases:

- (a) If  $E_m^\dagger E_p \in \mathcal{S}_{Q,I}$ , then  $E_m^\dagger E_p |\Psi_i\rangle = |\Psi_i\rangle$ . The errors have the same syndrome, but they also act on the code space the same way. (This is the case of a *degenerate* code.)
- (b) If  $E_m^\dagger E_p \in \mathcal{S}_{C,I}$ , then by (10),  $E_m^\dagger E_p |\Psi_i\rangle = \pm |\Psi_i\rangle$ . The errors have the same syndrome, but their effects differ by a possible global phase without changing the classical information  $i$  embedded in the codeword  $|\Psi_i\rangle$ . Therefore, we can still recover both the quantum and classical information. (See Theorem 3).
- (c) For all the rest, the errors act nontrivially on the codewords  $\{|\Psi_i\rangle\}$ , but do not have a unique syndrome. If this case applies to any pair of errors  $E_m, E_p \in \mathbf{E}$  then the error set  $\mathbf{E}$  is uncorrectable.

Combining these cases, we get that whenever  $E_m^\dagger E_p \in \langle \mathcal{S}_{Q,I}, \mathcal{S}_{C,I} \rangle \cup (\mathcal{G}^n - N(\mathcal{S}_Q)) \forall E_m, E_p \in \mathbf{E}$ , the codewords  $\{|\Psi_i\rangle\}$  can be recovered up to a possible global phase.  $\square$

**Theorem 3** *Once error recovery has been performed, the classical index  $i$  may be determined by measuring each of the  $g'_k \in \tilde{\mathcal{S}}_C$  observables. The original quantum state  $|\phi\rangle$  may be recovered by performing the unitary  $U_i^{-1}$  and then discarding the ancillae.*

**Proof** After we have performed error recovery, the state in our possession will be  $\pm |\Psi_i\rangle$ . Measuring the generator set  $\{g'_k\}$  of  $\tilde{\mathcal{S}}_C$  will guarantee proper identification of  $\mathbf{x}_i$  by (10). Once the classical index has been identified, we can see from (6) that we may recover the original quantum state  $|\phi\rangle$  by performing  $U_i^{-1}$  and discarding the states  $\pm |0\rangle^{\otimes s} |\Phi_+\rangle^{\otimes e}$ .  $\square$

## B. The Symplectic Formalism

In the following, we will use the symplectic formalism to formulate this problem and at the same time generalize Theorem 1. The goal here is to show that actually the EACQs can be completely specified by some classical parity check matrix  $H$  and quantum parity check matrix  $\hat{H}$ . Since an  $[[n, q; e]]$  EAQEC can be defined by a  $(s + 2e) \times 2n$  quantum parity check matrix  $\hat{F}$  as shown in (3), we may specify the quantum stabilizer  $\mathcal{S}_Q$  by  $\hat{F}$  and a binary matrix  $F$ :

$$F = \left( \begin{array}{cc|cc} \mathbf{0}_{s-c_1 \times c_1} & I_{s-c_1 \times s-c_1} & \mathbf{0}_{s-c_1 \times e-c_2} & \mathbf{0}_{s-c_1 \times c_2} \\ \mathbf{0}_{e-c_2 \times s-c_1} & \mathbf{0}_{e-c_2 \times c_1} & \mathbf{0}_{e-c_2 \times c_2} & I_{e-c_2 \times e-c_2} \\ \mathbf{0}_{e-c_2 \times s-c_1} & \mathbf{0}_{e-c_2 \times c_1} & \mathbf{0}_{e-c_2 \times e-c_2} & \mathbf{0}_{e-c_2 \times c_2} \end{array} \right), \quad (16)$$

where  $I_{r \times r}$  is the  $r \times r$  identity matrix, and  $\mathbf{0}_{r \times t}$  is the  $r \times t$  null matrix. That is,

$$\mathcal{S}_Q = \{N_{\mathbf{v}} | \mathbf{v} \in \text{Rowspace}(\hat{G})\}, \quad (17)$$

where  $\hat{G} = F\hat{F}$ .

Now, we may take any full rank,  $(s + 2e) \times (s + 2e)$  matrix  $M$  and write

$$F\hat{F} = (FM)(M^{-1}\hat{F}) = H\hat{H},$$

where  $H = FM$  and  $\hat{H} = M^{-1}\hat{F}$ . Since  $M$  is full rank,  $\text{Rowspace}(\hat{H}) = \text{Rowspace}(\hat{F})$ , and  $\hat{H}$  and  $\hat{F}$  specify the same stabilizer. However,  $H$  may be any  $(s + 2e - c) \times (s + 2e)$  matrix having linearly independent rows, so  $H$  is in fact an arbitrary classical parity-check matrix!

Although one can always use Theorem 1 to specify the code, it may be somewhat tedious to find the  $g'_k \in$

$\mathcal{S}_C$  in practice. Therefore, when formulating a code in the language of parity-check matrices, it may sometimes be more convenient to use a different set of eigenvalue equations to take advantage of our *a priori* knowledge of the properties of the classical parity-check matrix  $H$ .  $H$  specifies a set of  $2^c$  classical codewords  $\mathbf{y}_i \in (\mathbb{Z}_2)^{s+2e}$  satisfying

$$H\mathbf{y}_i^T = \mathbf{0}, i = 1, 2, \dots, 2^c. \quad (18)$$

**Theorem 4** *Assume we are given an  $(s + 2e) \times 2n$  quantum parity-check matrix  $\hat{H}$  with rows  $\mathbf{u}_l, l = 1, 2, \dots, (s + 2e)$ , and an  $(s + 2e - c) \times (s + 2e)$  classical parity-check matrix  $H$  whose kernel is  $\{\mathbf{y}_i\}, i = 1, 2, \dots, 2^c$ . Then we may fully specify the codewords by the equations,  $\forall i, l$ ,*

$$N_{\mathbf{u}_l} |\Psi_i\rangle = (-1)^{y_{il}} |\Psi_i\rangle. \quad (19)$$

**Proof** Theorem 1 can be rewritten as,  $\forall i, j$ ,

$$N_{\mathbf{u}_j} |\Psi_i\rangle = (-1)^{x_{ij}} |\Psi_i\rangle,$$

where  $\{\mathbf{x}_i\}$  is the kernel of  $F$ , and  $\mathbf{u}_j$  is the  $j$ -th row of  $\hat{F}$ . Since  $\hat{H} = M^{-1}\hat{F}$ , then

$$\begin{aligned} N_{\mathbf{u}_l'} |\Psi_i\rangle &= \prod_{m=1}^{s+2e} (N_{\mathbf{u}_m})^{(M^{-1})_{lm}} |\Psi_i\rangle, \\ &= (-1)^{\sum_{m=1}^{s+2e} (M^{-1})_{lm} x_{im}} |\Psi_i\rangle, \\ &= (-1)^{y_{il}} |\Psi_i\rangle, \end{aligned} \quad (20)$$

where  $\mathbf{y}_i = M^{-1}\mathbf{x}_i$ . In order to be valid codewords,  $\{|\Psi_i\rangle\}$  must also satisfy  $N_{\mathbf{w}_j} |\Psi_i\rangle = |\Psi_i\rangle$ , where  $\mathbf{w}_j$  is the  $j$ -th row of  $H\hat{H}$ . Then

$$\begin{aligned} N_{\mathbf{w}_j} |\Psi_i\rangle &= \left( \prod_{l=1}^{s+2e} (N_{\mathbf{u}_l'})^{H_{jl}} \right) |\Psi_i\rangle, \\ &= (-1)^{\sum_{l=1}^{s+2e} H_{jl} y_{il}} |\Psi_i\rangle, \\ &= (-1)^0 |\Psi_i\rangle = |\Psi_i\rangle. \end{aligned}$$

This concludes our proof.  $\square$

We have now established a new set of codewords with stabilizer

$$\mathcal{S}_Q = \left\{ N_{\mathbf{v}} | \mathbf{v} \in \text{Rowspace}(H\hat{H}) \right\},$$

and a new way of specifying the codewords via (19). Theorem 2 was cast in general enough terms that it is applicable to our new code. So we are now in a position to give the error-correcting conditions and to explain how to perform error detection and recovery in the language of the symplectic form as a corollary to Theorem 2.

**Corollary 5** *The set of correctable errors  $\mathbf{E}$  for a code specified by the quantum parity-check matrix  $\hat{H}$  and classical parity-check matrix  $H$  are such that for every distinct  $N_{\mathbf{e}}, N_{\mathbf{e}'} \in \mathbf{E}$ , either*

1.  $\mathbf{e} - \mathbf{e}' \in \text{Rowspace}(\hat{H}_I)$ , or
2.  $H\hat{H} \odot (\mathbf{e} - \mathbf{e}')^T \neq \mathbf{0}$ .

**Proof** Since

$$\langle \mathcal{S}_{Q,I}, \mathcal{S}_{C,I} \rangle = \{ N_{\mathbf{u}} | \mathbf{u} \in \text{Rowspace}(\hat{H}_I) \},$$

condition 1 corresponds to

$$N_{\mathbf{e}-\mathbf{e}'} = N_{\mathbf{e}}^\dagger N_{\mathbf{e}} \in \langle \mathcal{S}_{Q,I}, \mathcal{S}_{C,I} \rangle.$$

Let  $\mathbf{v}_j$  denote the  $j$ -th row of  $H\hat{H}$ ; condition 2 is equivalent to the statement that for  $\mathbf{e}$  and  $\mathbf{e}'$  there exists a  $\mathbf{v}_j$  such that

$$\left[ N_{\mathbf{e}'}^\dagger N_{\mathbf{e}}, N_{\mathbf{v}_j} \right] \neq 0$$

Therefore, conditions 1 and 2 together are equivalent to  $N_{\mathbf{e}'}^\dagger N_{\mathbf{e}} \in \langle \mathcal{S}_{Q,I}, \mathcal{S}_{C,I} \rangle \cup (\mathcal{G}^n - N(\mathcal{S}_Q))$ , which are the error correcting conditions of Theorem 2.  $\square$

### C. Properties of EACQs

**Theorem 6** *We can transform any  $[[n, q + c, d_1; e]]$  EAQECC code  $\mathcal{C}_1$  into an  $[[n, q : c, d_2; e]]$  EACQ code  $\mathcal{C}_2$ , and transform any  $[[n, q : c, d_2; e]]$  EACQ code  $\mathcal{C}_2$  into an  $[[n, q, d_3; e]]$  EAQECC code  $\mathcal{C}_3$ , where  $d_1 \leq d_2 \leq d_3$ .*

**Proof** The stabilizer group  $\mathcal{S}_Q$  of  $\mathcal{C}_1$  is of size  $2^{s+2e}$ , where  $s + q + c + e = n$ . The isotropic subgroup  $\mathcal{S}_{Q,I}$  and the symplectic subgroup  $\mathcal{S}_{Q,S}$  of  $\mathcal{S}_Q$  is of size  $2^s$  and  $2^{2e}$ , respectively. If we simply add an Abelian group  $\mathcal{S}_C$  of size  $2^c$  such that  $\mathcal{S}_C \cap \mathcal{S}_Q = \emptyset$ , then  $(\mathcal{S}_Q, \mathcal{S}_C)$  defines an  $[[n, q : c, d_2; e]]$  EACQ code  $\mathcal{C}_2$  for some  $d_2$ , which follows from Theorem 1. Let  $\mathbf{E}_1$  be the error set that can be corrected by  $\mathcal{C}_1$ , and  $\mathbf{E}_2$  be the error set that can be corrected by  $\mathcal{C}_2$ . Clearly,  $\mathbf{E}_1 \subset \mathbf{E}_2$  (see table I), so  $\mathcal{C}_2$  can correct more errors than  $\mathcal{C}_1$ . Therefore,  $d_2 \geq d_1$ .

In general, an  $[[n, q : c, d_2; e]]$  EACQ code  $\mathcal{C}_2$  is defined by  $\mathcal{S}_Q = \langle \mathcal{S}_{Q,I}, \mathcal{S}_{Q,S} \rangle$  and  $\mathcal{S}_C = \langle \mathcal{S}_{C,I}, \mathcal{S}_{C,S} \rangle$ , where the isotropic subgroup  $\mathcal{S}_{Q,I}$  and the symplectic subgroup  $\mathcal{S}_{Q,S}$  of  $\mathcal{S}_Q$  is of size  $2^{s-c_1}$  and  $2^{2(e-c_2)}$ , respectively, and the isotropic subgroup  $\mathcal{S}_{C,I}$  and the symplectic subgroup  $\mathcal{S}_{C,S}$  of  $\mathcal{S}_C$  is of size  $2^{c_1}$  and  $2^{2c_2}$ , respectively. Here the parameters satisfy  $s + q + e = n$  and  $c_1 + 2c_2 = c$ . Now let  $\mathcal{S}'_{Q,I} = \langle \mathcal{S}_{Q,I}, \mathcal{S}_{C,I} \rangle$  and  $\mathcal{S}'_{Q,S} = \langle \mathcal{S}_{Q,S}, \mathcal{S}_{C,S} \rangle$ . Then  $\mathcal{S}'_Q = \langle \mathcal{S}'_{Q,I}, \mathcal{S}'_{Q,S} \rangle$  defines an  $[[n, q, d_3; e]]$  EAQECC code  $\mathcal{C}_3$ . Let  $\mathbf{E}_3$  be the error set that can be corrected by  $\mathcal{C}_3$ . Let  $E \in \mathbf{E}_2$ , then either  $E \in \langle \mathcal{S}_{Q,I}, \mathcal{S}_{C,I} \rangle$  or  $E \notin N(\mathcal{S}_Q)$ .

- If  $E \in \langle \mathcal{S}_{Q,I}, \mathcal{S}_{C,I} \rangle$ , then  $E \in \mathcal{S}'_{Q,I}$ . Thus,  $E \in \mathbf{E}_3$ .
- Since  $\mathcal{S}_Q \subset \mathcal{S}'_Q$ , we have  $N(\mathcal{S}'_Q) \subset N(\mathcal{S}_Q)$ . If  $E \notin N(\mathcal{S}_Q)$ , then  $E \notin N(\mathcal{S}'_Q)$ . Thus,  $E \in \mathbf{E}_3$ .

Putting these together we get  $\mathbf{E}_2 \subset \mathbf{E}_3$ . Therefore  $d_3 \geq d_2$ .  $\square$

It is worth pointing out that the theory of EACQ codes naturally includes the set of classically enhanced quantum codes that do not require entanglement as a subclass. These would be codes for which there is no nontrivial symplectic subgroup for either  $\mathcal{S}_Q$  or  $\mathcal{S}_C$ , so that both of these groups are purely isotropic. In terms of the parameters describing the code, this is the special case where  $e = 0$ . Our first example in the next section is exactly such a code.

To conclude this section, we list the different error-correcting criteria of an EAQECC and an EACQ:

EAQECC	EACQ
$E_m^\dagger E_p \notin N(\langle \mathcal{S}_{Q,I}, \mathcal{S}_{Q,S} \rangle)$	$E_m^\dagger E_p \notin N(\langle \mathcal{S}_{Q,I}, \mathcal{S}_{Q,S} \rangle)$
$E_m^\dagger E_p \in \mathcal{S}_{Q,I}$	$E_m^\dagger E_p \in \langle \mathcal{S}_{Q,I}, \mathcal{S}_{C,I} \rangle$

TABLE I: The error-correcting conditions of EAQECCs and EACQs.

## IV. EXAMPLES

### A. $[[9, 1 : 3, 3; 0]]$ EACQ

We first give an example of a code that starts from an overly redundant quantum code, and exploits that redundancy by additionally encoding classical information. Starting from the 9-qubit Shor code, we modify it to encode three additional classical bits into the quantum code. The modified Shor code presented here encodes one qubit and three classical bits into nine physical qubits, and it is still able to correct an arbitrary error on a single qubit.

The code is a straightforward combination of the original 9 qubit Shor code, with parity-check matrix

$$\hat{H} = \left( \begin{array}{cccccccc|cccccccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array} \right),$$

and the  $[8, 3]$  classical code, with parity check matrix

$$H = \left( \begin{array}{cccccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \end{array} \right).$$

Table II gives the generators of  $\mathcal{S}_Q$  and  $\mathcal{S}_C$  as in (9) and (10) for the code.

$\mathcal{S}_Q$	$g_1$	Z	Z	I	Z	Z	I	Z	Z	I
	$g_2$	I	I	I	I	Z	Z	I	Z	Z
	$g_3$	Z	I	Z	Z	Z	I	I	I	I
	$g_4$	Y	Y	X	X	Y	Y	I	I	I
	$g_5$	Z	I	Z	Y	X	Y	Y	X	Y
$\mathcal{S}_C$	$g'_1$	Z	Z	I	I	I	I	I	I	I
	$g'_2$	I	Z	Z	I	I	I	I	I	I
	$g'_3$	I	I	I	I	Z	Z	I	I	I

TABLE II: The resulting  $[[9, 1 : 3, 3; 0]]$  EACQ encodes one qubit and three classical bits into nine physical qubits.

**Proposition 7** *The modified Shor code presented above can correct an arbitrary error on a single qubit.*

**Proof** This modified Shor code is degenerate. A single-qubit  $Z$  error on any of the qubits in the same triplet (that is, on any of qubits 1, 2, 3, or any of qubits 4, 5, 6,

or any of qubits 7, 8, 9) result in the same error syndrome, and can be corrected using the same recovery operation. However, each of the single-qubit  $X$  errors gives a distinct error syndrome, and can therefore be corrected. The syndromes are obtained by measuring  $\{g_1, \dots, g_5\}$ .  $\square$

### B. $[[8, 1 : 3, 3; 1]]$ EACQ code

The following example comes from modifying the  $[[8, 1, 3; 1]]$  EAQECC code given in [13]. The  $[[8, 1 : 3, 3; 1]]$  EACQ code comes from a combination of the  $[[8, 1, 3; 1]]$  EAQECC with the quantum parity check matrix

$$\hat{H} = \left( \begin{array}{cccccccc|cccccccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array} \right),$$

and the  $[8, 3]$  classical code, with parity check matrix

$$H = \left( \begin{array}{cccccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \end{array} \right).$$

$\hat{H}$  and  $H$  together specify  $(\mathcal{S}_Q, \mathcal{S}_C)$  for the EACQ given in Table III. The resulting EACQ encodes one qubit and

$\mathcal{S}_{Q,I}$	$g_1$	Z	Z	I	Z	Z	I	Z	Z	I
	$g_2$	Z	I	Z	Z	Z	I	I	I	I
	$g_3$	Y	Y	X	X	Y	Y	I	I	I
$\mathcal{S}_{Q,S}$	$g_4$	I	I	I	I	Z	Z	I	Z	Z
	$g_5$	Z	I	Z	Y	Y	X	Y	Y	Y
$\mathcal{S}_C$	$g'_1$	Z	Z	I	I	I	I	I	I	I
	$g'_2$	I	Z	Z	I	I	I	I	I	I
	$g'_3$	I	I	I	I	Z	Z	I	I	I

TABLE III: The resulting  $[[8, 1 : 3, 3; 1]]$  EACQ encodes one qubit and three classical bits into eight physical qubits with the help of one ebit.

three classical bits into eight physical qubits with the help of one ebit. Since the  $[[8, 1, 3; 1]]$  code is derived from the Shor code, this EACQ is clearly related to our first example.

### C. EACQ codes from classical BCH codes

Here, we will look at the  $[[63, 21, 9; 6]]$  EAQECC shown in [13], which is constructed from a classical binary  $[63, 39, 9]$  BCH code [17]. This EAQECC has the interesting property that removing the symplectic pairs from the quantum parity check matrix will only decrease the distance from  $d = 9$  to  $d = 7$  no matter how many pairs are removed. Therefore, if we switch all the ebits from  $\mathcal{S}_Q$  to  $\mathcal{S}_C$ , we will have a  $[[63, 21; 12, 7; 6]]$  EACQ. This example shows that it is possible to encode extra classical information using ebits without degrading the distance performance too much.

## V. CONCLUSIONS

In this paper, we have demonstrated yet another extension of the standard quantum error correction scheme. The new formalism, EACQ, is a quantum error-correcting code that can transmit both classical and quantum information simultaneously. We consider this EACQ formalism as both a generalization and a unification of EAQECCs and classical error correction, in the following sense:

- For a purely quantum code ( $c = 0$ ), we have  $\mathcal{S}_C = \emptyset$ . Then this corresponds to the entanglement-assisted formalism. In this case, the classical parity check matrix  $H$  is chosen to be

$$H = I_{(n-q) \times (n-q)}$$

such that the quantum parity-check matrix is  $\hat{G} = H\hat{H} = \hat{H}$  for the code.

- For a purely classical code ( $q = 0$ ), we have  $\mathcal{S}_Q = \emptyset$ . In this case, the quantum parity check matrix  $\hat{H}$  is

chosen to be

$$\hat{H} = (I_{n \times n} | \mathbf{0}_{n \times n})$$

such that the quantum parity-check matrix  $\hat{G} = H\hat{H} = (H | \mathbf{0}_{n \times n})$  for the code. The classical code can be thought of as encoded in the  $Z$  basis.

On the other hand, the EACQ formalism provides further flexibility in the use of quantum error correcting codes. As shown in the example section, the EACQ can make use of extra redundancy in quantum codes by encoding additional classical information. We also note that the passive error correcting ability of an EACQ is increased at the cost of the quantum code rate of an EAQECC.

We are currently investigating the relation between EACQs and other extensions of standard quantum error correction, such as OQECC or “operator algebra quantum error correction” (OAQEC) [3]. Recently we are aware of the work [4], which also allows correction of hybrid classical-quantum information based on operator algebra. Given the wider variety of resources in quantum information theory compared to classical information theory, we can expect a correspondingly richer set of families of quantum error-correcting codes.

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